PHYS213 Fall 2014 Midterm Solutions

7. Water, having a temperature of $10\ °C$, is placed into a pot made of stainless steel. The volume of water is 1 L. The diameter of the bottom of the pot is 20 cm. The thickness of the bottom is 5 mm. The thermal conductivity of the stainless steel is 16 $W\ m^{-1}K^{-1}$. Specific heat of water is $4187\ J\ kg^{-1}K^{-1}$. Take the density of water to be $1000\ kg/m^3$. We now place the pot on a hot plate of temperature $300\ °C$. How long do we have to wait for the temperature to change from $10\ °C$ to $11°C$?

Solution:

Area of cross-section for the bottom of the pot,

$$A = \frac{\pi d^2}{4} = 0.031\ m^2.$$

The amount of heat transferred to the water to raise its temperature from $10\ °C$ to $11°C$ is,

$$\Delta Q = \rho V c_{sp} \Delta T = 1000 * 1000^{-1} * 4187 * 1\ J = 4187\ J.$$

The rate of heat conduction is given by,

$$\frac{\Delta Q}{\Delta t} = \frac{\kappa A \Delta T}{L}$$

Which implies

$$\Delta t = \frac{\mathrm{L}\Delta Q}{\kappa A \Delta T} = \frac{0.005 * 4187}{16 * 0.031 * (300 - 10)} = \mathbf{0.145\ s}.$$

The next three problems refer to the following situation:

Consider a box that is split in two by a freely moving piston. The left side of the partition is filled with Helium gas and has a volume of 3 m^3 . The right side is filled with Nitrogen gas and has a volume of 2 m^3 . The two gases are in equilibrium at a temperature of 300K. There are 130 moles of Nitrogen gas. The barrier/piston is then removed.

8. What is the partial pressure due to the Nitrogen gas of the final mixture?

Solution:

The new volume occupied by the Nitrogen gas is 5 m^3, it's at a temperature of 300 K and there are 130 moles of it. According to the ideal gas equation of state:

$$p = \frac{nRT}{V} = \frac{130 * 8.314 * 300}{5} = \mathbf{64.85\ kPa}.$$

9. How much does the total entropy of the combined system increase due to the mixing? (Assume this is a dilute ideal gas.)

Solution:

Initially when the partition had not been removed, the two gasses were at the same pressure and temperature, which implies that the ratio of the number of moles of each gas present is the same as the ratio of the volumes occupied by them, i.e.,

$$\frac{n_{He}}{n_N} = \frac{V_{He}}{V_N} \Rightarrow n_{He} = \frac{V_{He}}{V_N} n_N = 1.5\, n_N.$$

Now, the number of available microstates for a gas depends on the volume V occupied by it as

$$\Omega = V^N,$$

Where N is the number of molecules of the gas present. The total number of microstates before mixing was then,

$$\Omega_{T,i} = \Omega_{He,i}\Omega_{N,i} = 3^{n_{He}N_a} \times 2^{n_N N_a},$$

Where $n_{He}N_a$ and $n_N N_a$ are the total number of molecules of Helium and Nitrogen respectively, and N_a is Avogadro's constant.

The entropy before the mixing was then,

$$S_i = k_B \ln \Omega_{T,i} = k_B\, n_N\, N_a(1.5 \ln 3 + \ln 2)\, J/K.$$

The number of available microstates after the mixing is

$$\Omega_{T,f} = \Omega_{He,f}\Omega_{N,f} = 5^{n_{He}N_a} \times 5^{n_N N_a} = 5^{N_a(n_{He}+n_N)}.$$

Thus the final entropy is

$$S_f = k_B \ln \Omega_{T,f} = k_B\, n_N\, N_a(1.5+1) \ln 5\;\; J/K.$$

The change in entropy then is,

$$\Delta S = S_f - S_i = k_B\, n_N\, N_a(2.5 \ln 5 - 1.5 \ln 3 - \ln 2)$$

$$= 1.38 * 10^{-23} * 130 * 6.022 * 10^{23} 1.68$$

$$= \mathbf{1817.71}\, \boldsymbol{J/K}$$

10. What is the probability that all the nitrogen gas returns to the side of the partition that it started in?

Solution:

The probability is given by the ratio of the number of microstates occupied by the gas when all of it is in the volume that was occupied by it before the mixing (2 m^3), to the total number of microstates available to it when it occupies the entire box (5 m^3), i.e.,

$$P = \frac{\Omega_{T,i}}{\Omega_{T,f}} = \frac{\Omega_{N,i}}{\Omega_{N,f}} = \frac{2^{n_N N_a}}{5^{n_N N_a}} = \left(\frac{2}{5}\right)^{130*6.022*10^{23}} = \left(\frac{\mathbf{2}}{\mathbf{5}}\right)^{\mathbf{783*10^{23}}}.$$

The next two problems refer to the following situation:

A particle is randomly hoping on a 1 dimensional lattice with spacing 3×10^{-2} m. Every 2.5×10^{-2} seconds it hops either to the left or to the right with equal probability.

11. After exactly 1 second what is the probability the particle will be at the same location that it started?

The position of the particle is given by,

$$(n_+ - n_-)a = x,$$

Where n_+ and n_- are the number of forward and backward steps respectively, a is the lattice spacing, and

$$n_+ + n_- = n,$$

where n is the total number of steps taken in 1 second, i.e. n=40. If the particle has to come back to its starting point, it must take as many forward steps as backwards, i.e.,

$$n_+ = n_- = \frac{n}{2},$$

Now the probability that out of n steps, the particle takes n_+ forward steps, and n_- backward steps, is given by the binomial distribution:

$$P(n_+) = \frac{n!}{n_+!\, n_-!} p^{n_+} q^{n_-},$$

Where p and q are the probabilities of taking steps forward and backwards respectively. Here p=q=0.5. Then the probability of the particle coming back to its starting point is just the probability that there are an equal number of forward and backward steps, i.e.,

$$P(x = 0) = P\left(n_+ = n_- = \frac{n}{2}\right) = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} p^{\frac{n}{2}} q^{\frac{n}{2}},$$

$$= \frac{40!}{20!\, 20!} (0.5)^{20} (0.5)^{20},$$

$$= 1.378 \times 10^{11} \times 0.5^{40},$$

$$= \mathbf{0.125}$$

12. Roughly estimate how long we would have to wait until the particle reaches the edge of the lattice whose total size is 2m assuming that the particle is released from the centre.

The average distance away from the centre after n steps is,

$$< x >=< (n_+ - n_-)a > = 0,$$

The average of the square of the distance from the centre after n steps is, however,

$$< x^2 \geq< (n_+ - n_-)^2 a^2 \geq n\, a^2.$$

This means that the distance away from the centre after n steps is given by $\sqrt{n}a$. If the particle starts at the centre, it has to travel a distance of 1m to reach the edge. Then the number of steps needed to reach the edge is,

$$n = \frac{< 1m^2 >}{a^2} = \frac{1}{0.03 \times 0.03} = 1111.11.$$

The time taken to reach the edge is then,

$$t = n * \Delta T = 1111.11 * 0.025 = \mathbf{27.78\ s}.$$